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# DIFFERENCE EQUATIONS IN DISCRETE SPACES

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**Abstract.** One considers a general difference equation with variable coefficients in discrete spaces. The conditions for unique solvability and Fredholmness for such equation are given using the theory of periodic Riemann boundary value problem. Key role in the studying takes the periodic analogue of the Hilbert transform, it permits to obtain explicit solution for particular cases. Also, this transform has very important properties related to a holomorphy. It leads to Fredholm properties for more general cases of difference equations.

**Keywords:** discrete spaces, Hilbert transform, general difference equation, Fredholm solvability

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# РАЗНОСТНЫЕ УРАВНЕНИЯ В ДИСКРЕТНЫХ ПРОСТРАНСТВАХ

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**Аннотация.** Рассматривается общее разностное уравнение с переменными коэффициентами в дискретных пространствах. Приведены условия однозначной разрешимости и фредгольмовости такого уравнения с использованием теории периодической краевой задачи Римана. Ключевую роль в изучении играет периодический аналог преобразования Гильберта, позволяющий получить явное решение для частных случаев. Кроме того, это преобразование обладает очень важными свойствами, связанными с голоморфностью. Это приводит к свойствам Фредгольма для более общих случаев разностных уравнений.

**Ключевые слова:** дискретные пространства, преобразование Гильберта, общее разностное уравнение, фредгольмова разрешимость

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**1. Introduction.** We consider a general linear difference equation of the type

$$\sum_{-\infty}^{+\infty} a_k(x)u(x+\beta_k) = v(x), \quad x \in D, \tag{1}$$

where $D$ is the space $\mathbf{R}^{\mathbf{m}}$ or a half-space $\mathbf{R}^m_+$ for a continual case, and $\mathbf{Z}^m$ and discrete half-space $\mathbf{Z}^m_+$ for a discrete one, $\{\beta_k\} \subset D$ is given sequence, $\beta_k = (\beta_{k_1}, \cdots, \beta_{k_m})$. For continual variable $x$ we use the term "difference equation and for discrete one "discrete equation".

Situations are very distinct if we consider the equation on a whole space or on a half-space. Here we'll consider the case $\mathbf{Z}^m_+$ because other situations will be considered in separate publications.

Such equations arise in many applied problems, for example in a control theory and digital signal processing [1], thus a problem of their solvability is a very actual. We choice the space $L_2(D)$ as an initial functional space, but these equations can be considered in more general spaces $L_p(D)$. Key point in our study plays the special lacunary Fourier series

$$\sigma(x,\xi)=\sum_{-\infty}^{+\infty}a_k(x)e^{i\beta_k\cdot\xi}, \tag{2}$$

assuming the series (2) is convergent almost everywhere.

**Definition 1.** *The function $\sigma(x,\xi)$ is called a symbol of the equation* (1) *or a symbol of the discrete-difference operator*

$$\mathcal{D}:\ u(x)\longmapsto\sum_{-\infty}^{+\infty}a_k(x)u(x+\beta_k),\quad x\in D.$$

**Remark.** This function $\sigma(x,\xi)$ is a periodic function on variable $\xi$ because $\{\beta_k\}\subset\mathbf{Z}^m$. We'll denote its basic cube of periods by $[-T,T]^m$ and if $\{\beta_k\}=\mathbf{Z}^m$ then $T=\pi$.

**2. Difference equations with constant coefficients.** The first step in our studying will be the following discrete equation with constant coefficients

$$\sum_{|k|=0}^{+\infty}a_ku(\tilde{x}+\beta_k)=v(\tilde{x}),\quad \tilde{x}\in\mathbf{Z}^m_+,\quad \{\beta_k\}\subset\mathbf{Z}^m_+, \tag{3}$$

or in other words finding invertibility conditions for the operator

$$\mathcal{D}_{\tilde{x}_0}:\ u(\tilde{x})\longmapsto\sum_{|k|=0}^{+\infty}a_k(\tilde{x}_0)u(\tilde{x}+\beta_k),\quad \tilde{x}\in\mathbf{Z}^m_+, \tag{4}$$

where the point $\tilde{x}_0\in\mathbf{Z}^m_+$ is fixed.

Given operator (4) one defines its symbol

$$\sigma(x_0,\xi)=\sum_{-\infty}^{+\infty}a_k(\tilde{x}_0)e^{i\beta_k\cdot\xi},$$

and introduces.

**Definition 2.** *A symbol $\sigma(x,\xi)$ is called an elliptic symbol if $\sigma(\tilde{x},\xi)\neq 0,\ \forall\tilde{x}\in\mathbf{Z}^m_+,\ \xi\in[-T,T]^m$.*

Further one considers a more general equation with two difference-discrete operators $\mathcal{A},\mathcal{B}$ with constant coefficients and two projectors $P_\pm$ on a discrete half-space $\mathbf{Z}^m_\pm$. More precisely let's denote

$$\mathcal{A}:\ u(\tilde{x})\longmapsto\sum_{|k|=0}^{+\infty}a_ku(\tilde{x}+\alpha_k),\ \mathcal{B}:\ u(\tilde{x})\longmapsto\sum_{|k|=0}^{+\infty}b_ku(\tilde{x}+\beta_k),$$

$$\tilde{x}\in\mathbf{Z}^m_+,\ \{\alpha_k\},\{\beta_k\}\subset\mathbf{Z}^m_+,$$

and consider the equation

$$(\mathcal{A}P_++\mathcal{B}P_-)U=V \tag{5}$$

in the space $L_2(\mathbf{Z}^m)$. We denote symbols of operators $\mathcal{A},\mathcal{B}$ by $\sigma_{\mathcal{A}}(\xi),\sigma_{\mathcal{B}}(\xi)$.

It is well known the equation (3) is equivalent to the equation (5) with $\mathcal{B}=I, I$ is an identity operator, that's why we study the equation (5).

**2.1. Periodic analogue of the Hilbert transform.** We denote $\xi=(\xi',\xi_m),\ \xi'=(\xi_1,\cdots,\xi_{m-1})$ and introduce two operators acting in the space $L_2([-\pi,\pi]^m)$

$$P^{per}_{\xi'}=\frac{1}{2}(I+H^{per}_{\xi'}),\quad Q^{per}_{\xi'}=\frac{1}{2}(I-H^{per}_{\xi'}),$$

where $H^{per}_{\xi'}$ is the following periodic analogue of the Hilbert transform

$$(H^{per}_{\xi'}u)(\xi',\xi_m)=\frac{1}{2\pi i}v.p.\int_{-\pi}^{+\pi}\cot\frac{\xi_m-\eta_m}{2}u(\xi',\eta_m)d\eta_m.$$

All details related to these operators can be found in authors' papers [14, 16]. Here we give some needed results only.

**Lemma 1.** *We have the following relations*

$$FP_+=P^{per}_{\xi'}F,\quad FP_-=Q^{per}_{\xi'}F.$$

Thus after applying the Fourier transform to (5) we obtain

$$\sigma_{\mathcal{A}}(\xi)(P^{per}_{\xi'}\tilde{U})(\xi)+\sigma_{\mathcal{B}}(\xi)(Q^{per}_{\xi'}\tilde{U})(\xi)=\tilde{V}(\xi), \tag{6}$$

and the last equation (6) is one-dimensional characteristic singular integral equation with the Hilbert kernel depending on a parameter $\xi'$ [3, 8]. But for solving this equation we need another periodic analogue of the Riemann boundary value problem than usual Riemann-Hilbert problem [3, 8].

**2.2. Periodic Riemann boundary value problem.** We formulate this problem in the following way [16]. Let $\Pi_\pm$ be two half-strips $\Pi_\pm=\{z\in\mathbf{C}: z=t+is, t\in[-\pi,\pi], \pm s>0\}$.

Periodic Riemann boundary value problem is called the following problem [16]: finding a pair of functions $\Phi^\pm(z)$ analytical in $\Pi_\pm$ for which their boundary values under $s\to 0\pm$ satisfy on the segment $[-\pi;\pi]$ the following linear relation

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t),\quad t\in[-\pi;\pi],$$

where $G(t),\ g(t)$ are given functions on $[-\pi;\pi]$, $G(-\pi)=G(\pi),\ g(-\pi)=g(\pi)$.

For the solution one introduces an integral of the type

$$\Phi(z)=\frac{1}{4\pi i}\int\limits_{-\pi}^{\pi}\varphi(x)\cot\frac{x-z}{2}dx,\quad z\in\Pi_\pm,$$

which is analogue of the well known Cauchy type integral. Further we have analogue of Plemelj–Sokhotskii formulas [3, 8, 16]

**Lemma 2.** *If $\varphi(t)$ satisfies the Hölder condition on the segment $[-\pi;\pi]$, $\varphi(-\pi)=\varphi(\pi)$, then $(\Phi(z)$ has boundary values $\Phi^\pm(t)$ under $s\to 0\pm$ which are given by formulas*

$$\Phi^+(\tau)=\frac{1}{4\pi i}\int\limits_{-\pi}^{\pi}\varphi(t)\cot\frac{t-\tau}{2}dt+\frac{\varphi(\tau)}{2}+C,$$

$$\Phi^-(\tau)=\frac{1}{4\pi i}\int\limits_{-\pi}^{\pi}\varphi(t)\cot\frac{t-\tau}{2}dt-\frac{\varphi(\tau)}{2}+C;$$

*where the integral is treated in principal value sense.*

These formulas guarantee a validity of lemma 1 at least for smooth functions.

For solving the periodic Riemann boundary value problem and taking into account a paremeter $\xi'$ we give the following

**Definition 3.** *Factorization of the elliptic symbol $\sigma(\xi)$ on the variable $\xi_m$ is called its representation in the form*

$$\sigma(\xi',\xi)=\sigma_+(\xi',\xi)\cdot\sigma_-(\xi',\xi),$$

*where the factors $\sigma_\pm$ admit an analytical continuation into complex half-strips $\Pi_\pm$ for almost all fixed $\xi'\in[-T,T]^{m-1}$ and $\sigma_\pm\in L_\infty[-T,T]^m$.*

Such factorization and related constructions can be realized with help of the periodic Hilbert transform $H^{per}_{\xi'}$ in dependence of so called index of factorization [3, 8, 6, 4, 14, 16].

Assuming that $\sigma_{\mathcal{A}},\sigma_{\mathcal{B}}$ are continuous functions on $[-T,T]^m$ we fix $\xi'\in[-T,T]^{m-1}$ and define

$$æ(\xi')\equiv Ind\,\sigma=\frac{1}{2\pi}\int\limits_{-T}^{+T}d\arg(\sigma^{-1}_{\mathcal{A}}(\cdot,\xi_m)\sigma_{\mathcal{B}}(\cdot,\xi_m)).$$

This index is an integer, and indeed it doesn't depend on $\xi'$ if $m\geq 2$ (homotopy property). The case $m=1$ is a very specific one (see [17]). So we have

$$æ(\xi')=æ.$$

Now we are ready to formulate a basic result on unique solvability of the equation (5).

**Theorem 1.** *let $\sigma_{\mathcal{A}}(\xi),\sigma_{\mathcal{B}}(\xi)$ be elliptic symbols which are continuous on $[-T,T]^m$. The equation (5) is a uniquely solvable in the space $L_2(\mathbf{Z}^m)$ for arbitrary right hand side $V\in L_2(\mathbf{Z}^m)$ iff $æ=0$.*

**Proof.** The equation (6) can be rewritten as the Riemann boundary value problem

$$(P^{per}_{\xi'}\tilde{U})(\xi',\xi_m)=-\sigma^{-1}_{\mathcal{A}}(\xi',\xi_m)\sigma_{\mathcal{B}}(\xi',\xi_m)(Q^{per}_{\xi'}\tilde{U})(\xi',\xi_m)+ \sigma^{-1}_{\mathcal{A}}(\xi',\xi_m)\tilde{V}(\xi',\xi_m),\quad \xi_m\in[-\pi,\pi], \tag{7}$$

or as the one-dimensional singular integral equation

$$\frac{\sigma_{\mathcal{A}}(\xi',\xi_m)+\sigma_{\mathcal{B}}(\xi',\xi_m)}{2}\tilde{U}(\xi',\xi_m)+$$
$$\frac{\sigma_{\mathcal{A}}(\xi',\xi_m)-\sigma_{\mathcal{B}}(\xi',\xi_m)}{2}(H^{per}_{\xi'}\tilde{U})(\xi',\xi_m)=\tilde{V}(\xi',\xi_m),\quad \xi_m\in[-\pi,\pi], \tag{8}$$

with a parameter $\xi'\in[-\pi,\pi]^{m-1}$.

As it was shown in [16] if æ $=0$ the factorization on variable $\xi_m$ for the symbol $\sigma(\xi',\xi_m)\equiv-\sigma^{-1}_{\mathcal{A}}(\xi',\xi_m)\sigma_{\mathcal{B}}(\xi',\xi_m)$ and it can be constructed as follows

$$\sigma(\xi',\xi_m)=\exp(\Gamma^+(\xi',\xi_m))\exp(\Gamma^-(\xi',\xi_m)),$$

where

$$\Gamma^+(\xi',\xi_m)=P^{per}_{\xi'}(\ln\sigma(\xi',\xi_m)),\quad \Gamma^-(\xi',\xi_m)=Q^{per}_{\xi'}(\ln\sigma(\xi',\xi_m)).$$

If æ $\neq 0$ then there are either additional summands in a general solution or additional conditions on a right hand side. So the unique solvability for the equation (5) is possible only if æ $=0$. ■

**Note.** Of course last cases when the index is not zero are very important, and we hope to study them in forthcoming papers. One-dimensional constructions for such situations are described in [17].

**3. Variable coefficients and a Fredholm property.** Now we consider the operator

$$\mathcal{D}=\mathcal{A}P_+ + \mathcal{B}P_-$$

in the space $L_2(\mathbf{Z}^m)$ assuming that symbols $\sigma_{\mathcal{A}}(\tilde{x},\xi),\sigma_{\mathcal{B}}(\tilde{x},\xi)$ depend on a space discrete variable $\tilde{x}\in\mathbf{Z}^m$.

**3.1. Boundedness of difference operators.**

**Lemma 3.** *If*

$$\sum_{|k|=0}^{\infty}|a_k(\tilde{x})|<+\infty,\qquad \forall\tilde{x}\in\mathbf{Z}^m,$$

*then the operator $\mathcal{D}$ is a linear bounded operator $L_2(\mathbf{Z}^m)\to L_2(\mathbf{Z}^m)$ and its symbol $\sigma_{\mathcal{D}}(\tilde{x},\xi)$ is a bounded function defined on $\mathbf{Z}^m\times\mathbf{T}^m$.*

**Remark 2.** *It is more convenient to formulate a boundedness condition as a property of a symbol. So obviously if the symbol $\sigma_{\mathcal{D}}(\tilde{x},\xi)$ is a continuous function on $\dot{\mathbf{Z}}^m\times\mathbf{T}^m$ then the operator $\mathcal{D}$ is bounded. Here $\dot{\mathbf{Z}}^m$ denotes $\mathbf{Z}^m+\{\infty\}$.*

**3.1. Fredholmness.**

**Definition 4.** *Operator $\mathcal{D}$ is called a Fredholm operator if*

$$Ind\ \mathcal{D}\equiv\dim Ker\mathcal{D}-\dim Coker\mathcal{D}\neq\infty.$$

To move to more concrete results we need a one additional assumption on behavior of the symbol $\sigma_{\mathcal{D}}(\tilde{x},\xi)$. Let's fix $\tilde{x}\in\dot{\mathbf{Z}}^m$, $\xi'\in\mathbf{T}^{m-1}$ and consider the number

$$æ(\tilde{x},\xi')=\frac{1}{2\pi}\int_{-T}^{T}d\arg\sigma_{\mathcal{D}}(\tilde{x},\xi',\xi_m),$$

which is a winding number of the curve on a complex plane generated by $\sigma_{\mathcal{D}}(\tilde{x},\xi',\xi_m)$ when $\xi_m$ varies on the segment $[-T,T]$ [3, 8].

**Lemma 4.** *The number $æ(\tilde{x},\xi')=æ(\tilde{x})$ is an integer non-depending on $\xi'$.*

**Proof.** Indeed, under fixed $\tilde{x}\in\dot{\mathbf{Z}}^m$ this $æ(\tilde{x},\xi')$ is an integer valued function continuously depending on $\xi'\in[-T,T]^{m-1}$. Thus it takes the same values for all points $\xi'$. ■

**Definition 5.** *A local index of the operator $\mathcal{D}$ is called the number $æ(\tilde{x})$ whicn is defined for all $\tilde{x}\in\dot{\mathbf{Z}}^m$.*

**Remark 3.** This definition and properties of the symbol imply only that this local index can take only finite number of integer values on finite number of non-inersecting finite sets.

To obtain more applicable result we need an additional

**Assumption.** *We suppose that $\sigma_{\mathcal{D}}(\tilde{x},\xi)$ is a restriction of a continuous function defined on $\dot{\mathbf{R}}^m$, i.e. $\forall\sigma_{\mathcal{D}}(\tilde{x},\xi)\ \exists\,\sigma(x,\xi)\in C(\dot{\mathbf{R}}^m\times[-T,T^m])$ such that*

$$\sigma_{\mathcal{D}}(\tilde{x},\xi)=\sigma(\tilde{x},\xi),\quad \forall\tilde{x}\in\dot{\mathbf{Z}}^m.$$

**Lemma 5.** *Under assumption 1 the local index $æ(\tilde{x})$ doesn't depend on $\tilde{x}\in\dot{\mathbf{Z}}^m$:*

$$æ(\tilde{x})=æ.$$

**Proof.** Analogous considerations like lemma 4. ■

Definition 6. *The operator* $\mathcal{D}_{\tilde{x}_0} = \mathcal{A}_{\tilde{x}_0} P_+ + \mathcal{B}_{\tilde{x}_0} P_-$ *is called a local representative of the operator* $\mathcal{D}$ *in the point* $\tilde{x}_0 \in \mathbf{Z}^m$.

Lemma 6. *The operator* $\mathcal{D}$ *has a Fredholm property in the space* $L_2(\mathbf{Z}^m)$ *iff all its local representatives are invertible in the space* $L_2(\mathbf{Z}^m)$ *.*

**Sketch of proof.** Operators of type $\mathcal{D}$ are included in more wide set of operators. This set consists of operators of the type

$$AP_+ + BP_-,$$

where $A, B$ are pseudo differential operators with symbols $\sigma_A(\tilde{x}, \xi), \sigma_B(\tilde{x}, \xi), \tilde{x} \in \dot{\mathbf{Z}}^m \times [-T, T]^m$. Such operators can be defined by the formula

$$(Au)(\tilde{x}) = F^{-1}_{\xi \to \tilde{x}}(\sigma(\tilde{x}, \xi)\tilde{u}(\xi)), \tag{9}$$

where $F^{-1}_{\xi \to \tilde{x}}$ denotes a passing to the "Fourier coefficients"

$$(F^{-1}_{\xi \to \tilde{x}}\tilde{u})(\tilde{x}) = \frac{1}{(2\pi)^m} \int_{-\pi}^{\pi} \tilde{u}(\xi) e^{i\tilde{x}\cdot\xi} d\xi, \quad \tilde{x} \in \mathbf{Z}^m.$$

These operators are operators of local type [6] and can be reconstructed by their local representatives up to compact operator. Then using properties of Fredholm operators we obtain the needed assertion. ■

Theorem 2. *For the elliptic operator* $\mathcal{D}$ *to be a Fredholm operator in the space* $L_2(\mathbf{Z}^m)$ *it is necessary and sufficient to have*

$$æ = 0.$$

**Proof.** It is easy imlication from lemma 6 and results of section 2. ■

**Conclusion.** It seems these results can be useful and applicable for some concrete equations. Some unconsidered here questions will be discussed elsewhere and may be the object of another paper.

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