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THE CRITERION FOR THE UNIQUE SOLVABILITY OF THE DIRICHLET AND POINCARE SPECTRAL PROBLEMS FOR THE MULTIDIMENSIONAL EULER – DARBOUX – POISSON EQUATION

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Abstract. In the cylindrical region of Euclidean space for the multi-dimensional Euler – Darbu – Poisson equation, the spectral problems of Dirichle and Poincare are considered. The solution is sought in the form of decomposition by multidimensional spherical functions. The theorem of existence and uniqueness of the classical solution has been proved. Conditions of unique solvability of the assigned tasks are obtained, which depend significantly on the height of the cylinder.

Key words: Criteria, Spectral Problems, Multidimensional Equation, Cylindrical Domain, Bessel Function.

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КРИТЕРИЙ ОДНОЗНАЧНОЙ РАЗРЕШИМОСТИ СПЕКТРАЛЬНЫХ ЗАДАЧ ДИРИХЛЕ И ПУАНКАРЕ ДЛЯ МНОГОМЕРНОГО УРАВНЕНИЯ ЭЙЛЕРА – ДАРБУ – ПУАССОНА

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Аннотация. В цилиндрической области евклидова пространства для многомерного уравнения Эйлера – Дарбу -– Пуассона рассматриваются спектральные задачи Дирихле и Пуанкаре. Решение ищется в виде разложения по многомерным сферическим функциям. Доказаны теоремы существования и единственности классического решения. Получены условия однозначной разрешимости поставленных задач, которые существенно зависят от высоты цилиндра.

Ключевые слова: критерий, спектральные задачи, многомерное уравнение, цилиндрическая область, функция Бесселя.

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1. Introduction Two-dimensional spectral problems for hyperbolic equations are extensively studied (see for example [Kalmenov, 1993; Moiseev, 1988; Sabito, 2000; He K. Ch. 2000], and their multivariate analogues are studied in [Aldashev, 2003; Aldashev, 2005; Aldashev, 2006; Aldashev, 2014]. This is because three or more independent variables have difficulties of a fundamental nature. There is a highly attractive and convenient method of singular integral equations. Applied for two-dimensional problems, it cannot be used in virtue of absence of complete theory of multidimensional singular integral equations. The theory of multidimensional spherical functions, by contrast, is quite fully studied. These functions have important applications in mathematical physics, in theoretical physics, and in the theory of multidimensional singular integral equations. The author proposes that in solving the spectral problems of Dirichle and Poincare for the multidimensional Euler-Darbu-Poisson equation, the decomposition by spherical functions should be used.

2. Statement of the problem and result. Let $\Omega_\beta-$ be the cylindrical region of the Euclidean space E_{m+1} points $(x_1,...,x_m,t)$, bounded by the cylinder $\Gamma=\{(x,t):|x|=1\}$, by the planes $t=\beta>0$ and $t=0$, of where $|x|$ is a the length of the vector $x=(x_1,...,x_m)$. The parts of these surfaces that form the boundary $\partial\Omega_\beta$ of the domain Ω_β, are denoted by Γ_β, S_β, S_0 respectively. In the region Ω_β we consider the multidimensional Euler-Darboux-Poisson equation with the spectral real parameter γ

$$\Delta_x u-u_{tt}-\frac{\alpha}{t}u_t=\gamma u, \tag{1}$$

where Δ_x is the Laplacian operator with respect to the variables $x_1,...,x_m$, $m\geq 2$, $\alpha-$ and a is a real number.

By $u_\alpha(x,t)$ we denote the solution of equation (1) for given α.

As multidimensional Dirichlet and Poincare problems, we consider the following problems.

Problem 1. Find a solution to equation (1) in the region Ω_β from the class $C(\overline{\Omega}_\beta\setminus S_0)\cap C^2(\Omega_\beta)$, satisfying the boundary conditions

$$u_\alpha\Big|_{S_0}=0,\quad u_\alpha\Big|_{\Gamma_\beta}=0,\, u_\alpha\big|_{S_\beta}=0,\, \alpha<1; \tag{2}$$

$$\frac{u_\alpha}{\ln t}\Big|_{S_0}=0,\quad u_\alpha\Big|_{\Gamma_\beta}=0,\, u_\alpha\big|_{S_\beta}=0,\, \alpha=1; \tag{3}$$

$$(t^{\alpha-1}u_\alpha)\Big|_{S}=0,\quad u_\alpha\Big|_{\Gamma_\beta}=0,\, u_\alpha\big|_{S_\beta}=0,\, \alpha>1. \tag{4}$$

Problem 2. Find a solution to equation (1) in a domain Ω_β from the class $C(\overline{\Omega}_\beta\setminus S_0)\cap C^2(\Omega_\beta)$, satisfying the boundary conditions

$$\frac{\partial u_\alpha}{\partial t}\Big|_{S_0}=0,\quad u_\alpha\Big|_{\Gamma_\beta}=0,\, u_\alpha\big|_{S_\beta}=0,\, \alpha\geq 0; \tag{5}$$

$$\lim_{t\to 0}t^\alpha(u_\alpha-u_{\alpha,1})=0,\quad u_\alpha\Big|_{\Gamma_\beta}=0,\, u_\alpha\big|_{S_\beta}=0,\, \alpha<0, \tag{6}$$

where $u_{\alpha,1}(x,t)$ is the solution of the Cauchy problem for equation (1) with data $u_{\alpha,1}(x,0)=\tau(x)$, $\frac{\partial}{\partial t}u_{\alpha,1}(x,0)=0$.

Further, it is convenient for us to move from the Cartesian coordinates $x_1,...,x_m,t$ to spherical $r,\theta_1,...,\theta_{m-1},t$, $r\geq 0$, $0\leq\theta_1<2\pi$, $0\leq\theta_i\leq\pi$, $i=2,3,...,m-1$.

Let $\left\{Y_{n,m}^k(\theta)\right\}$ be a system of linearly independent spherical functions of order n, $1\leq k\leq k_n$, $(m-2)!n!k_n=(n+m-3)!(2n+m-2)$, $\theta=(\theta_1,...,\theta_{m-1})$.

Then the following result is valid.

Theorem. 1) *If $\gamma\leq-\mu_{s,n}^2$, then for all α problems 1 and 2 have only zero solutions.*

2) *If $\alpha\leq 0$ or $\alpha\geq 2$, then for $\gamma>-\mu_{s,n}^2$ problem 1 has only a trivial solution, if and only if*

$$\sin\beta\sqrt{\gamma+\mu_{s,n}^2}\neq 0,\ s=1,2,\ldots. \tag{7}$$

3) *For $0<\alpha<2$ and $\gamma>-\mu_{s,n}^2$ problem 1 has only a zero solution if and only if, the condition*

$$\cos\beta\sqrt{\gamma+\mu_{s,n}^2}\neq 0,\ \ s=1,2,\ldots. \tag{8}$$

4) *The solution of Problem 2 for $\gamma>-\mu_{s,n}^2$ for any α is only trivial if and only if relation (8) holds, where $\mu_{s,n}$ are positive zeros of the Bessel functions of the first kind $J_{n+\frac{(m-2)}{2}}(z)$.*

We note that for $\alpha=0$ this theorem was obtained in [Aldashev,2010; Aldashev, 2011].

3. Information of tasks 1 and 2 to two-dimensional problems. In spherical coordinates, the equation (1) has the form

$$u_{rr}+\frac{m-1}{r}u_r-\frac{1}{r^2}\delta u-u_{tt}-\frac{\alpha}{t}u_t=\gamma u, \tag{9}$$

$$\delta\equiv-\sum_{j=1}^{m-1}\frac{1}{g_j\sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\frac{\partial}{\partial\theta_j}\right),\ g_1=1,\ g_j=(\sin\theta_1...\sin\theta_{j-1})^2,\ j>1.$$

It is well known [Mikhlin, 1962], that the spectrum of the operator δ consists of eigenvalues $\lambda_n=n(n+m--2)$, $n=0,1,\ldots$, each of which corresponds to kn orthonormal eigenfunctions $Y_{n,m}^k(\theta)$.

Since the desired solutions to problems 1 and 2 belong to the class $C^2(\Omega_\beta)$, they can be sought in the form of a series

$$u_\alpha(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\bar{u}^k_{\alpha,n}(r,t)Y^k_{n,m}(\theta), \tag{10}$$

where $\bar{u}^k_{\alpha,n}(r,t)$ are functions to be determined.

Substituting (10) into (9), using the orthogonality of the spherical functions $Y^k_{n,m}(\theta)$ [Mikhlin, 1962], we obtain

$$L_\alpha\bar{u}^k_{\alpha,n}=\bar{u}^k_{\alpha,nrr}+\frac{m-1}{r}\bar{u}^k_{\alpha,nr}-\bar{u}^k_{\alpha,ntt}-\frac{\alpha}{t}\bar{u}^k_{\alpha,nt}-\frac{\lambda_n}{r^2}\bar{u}^k_{\alpha,n}-\gamma\bar{u}^k_{\alpha,n}=0,\ k=\overline{1,k_n},\ n=0,1,...,$$

which, using the substitution $\bar{u}^k_{\alpha,n}(r,t)=r^{\frac{(1-m)}{2}}\bar{u}^k_{\alpha,n}(r,t)$ reduces to the equation

$$L_\alpha u^k_{\alpha,n}=u^k_{\alpha,nrr}-u^k_{\alpha,ntt}-\frac{\alpha}{t}u^k_{\alpha,nt}+\frac{\bar{\lambda}_n}{r^2}u^k_{\alpha,n}-\gamma\bar{u}^k_{\alpha,n}=0,\ k=\overline{1,k_n},\ n=0,1,..., \tag{11_α}$$

$$\bar{\lambda}_n=\frac{(m-1)(3-m)-4\lambda_n}{4}.$$

Further, from the boundary conditions (2)–(6) for the functions $u^k_{\alpha,n}(r,t)$ by virtue of (9), we respectively have

$$u^k_{\alpha,n}(r,0)=0,\ u^k_{\alpha,n}(1,t)=0,\ u^k_{\alpha,n}(r,\beta)=0,\ \alpha<1,\ k=\overline{1,k_n},\ n=0,1,..., \tag{12}$$

$$\left.\frac{u^k_{\alpha,n}}{\ln t}\right|_{t=0}=0,\ u^k_{\alpha,n}(1,t)=0,\ u^k_{\alpha,n}(r,\beta)=0,\ \alpha=1,\ k=\overline{1,k_n},\ n=0,1,..., \tag{13}$$

$$\left.(t^{\alpha-1}u^k_{\alpha,n})\right|_{t=0}=0,\ u^k_{\alpha,n}(1,t)=0,\ u^k_{\alpha,n}(r,\beta)=0,\ \alpha>1,\ k=\overline{1,k_n},\ n=0,1,..., \tag{14}$$

$$\left.\frac{\partial u^k_{\alpha,n}}{\partial t}\right|_{t=0}=0,\ u^k_{\alpha,n}(1,t)=0,\ u^k_{\alpha,n}(r,\beta)=0,\ \alpha\geq 0,\ k=\overline{1,k_n},\ n=0,1,..., \tag{15}$$

$$\lim_{t\to 0}t^\alpha(u^k_{\alpha,n}-u^{k,1}_{\alpha,n})_t=0,\quad u^k_{\alpha,n}(1,t)=0,\ u^k_{\alpha,n}(r,\beta)=0,\ \alpha<0,\ k=\overline{1,k_n},\ n=0,1,\ldots. \tag{16}$$

In this way, problems 1 and 2 are reduced to two-dimensional spectral Dirichlet and Poincare problems for equation (11_α). The solution to these problems will be studied in sections 4 and 5.

Along with equation (11_α) we consider the equation

$$L_0u^k_{0,n}\equiv u^k_{0,nrr}-u^k_{0,ntt}+\frac{\bar{\lambda}_n}{r^2}\bar{u}^k_{0,n}-\gamma\bar{u}^k_{0,n}=0, \tag{11_0}$$

which, using the change of variables $\xi=\frac{r+t}{2}$, $\eta=\frac{r-t}{2}$ reduces to the equation

$$Mu^k_{0,n}\equiv u^k_{0,n\xi\eta}+\frac{\bar{\lambda}_n}{(\xi+\eta)^2}u^k_{0,n}=\gamma\bar{u}^k_{0,n}. \tag{17}$$

Solution of the Cauchy problem for (17) with data'

$$u^k_{0,n}(\xi,\xi)=\tau^k_n(\xi),\ \left.\left(\frac{\partial u^k_{0,n}}{\partial\xi}-\frac{\partial u^k_{0,n}}{\partial\eta}\right)\right|_{\xi=\eta}=\nu^k_n(\xi),\ 0\leq\xi\leq\frac{1}{2}$$

has the form [Aldashev, 1991].

$$\begin{aligned}u^k_{0,n}(\xi,\eta)&=\frac{1}{2}\tau^k_n(\eta)R(\eta,\eta;\xi,\eta)+\frac{1}{2}\tau^k_n(\xi)R(\xi,\xi;\xi,\eta)+\frac{1}{\sqrt{2}}\int_\eta^\xi[\nu^k_n(\xi_1)R(\xi_1,\xi_1;\xi,\eta)-\\&-\tau^k_n(\xi_1)\frac{\partial}{\partial N}R(\xi_1,\eta_1;\xi,\eta)|_{\xi_1=\eta_1}]d\xi_1+\gamma\int_{\frac{1}{2}}^{\xi}\int_0^{\eta}u^k_{0,n}(\xi_1,\eta_1)R(\xi_1,\eta_1;\xi,\eta)d\xi_1d\eta_1,\end{aligned} \tag{18}$$

where $R(\xi_1,\eta_1;\xi,\eta)=P_\mu[\frac{(\xi_1-\eta_1)(\xi-\eta)+2(\xi\eta+\xi_1\eta_1)}{(\xi_1+\eta_1)(\xi+\eta)}]=P_\mu(z)$ is the Riemann function for the equation $Mu^k_{0,n}=0$ [Copson, 1958], $P_\mu(z)$ is the Legendre function, $\mu=n+\frac{(m-3)}{2}$,

$$\left.\frac{\partial}{\partial N}\right|_{\xi=\eta}=\frac{1}{\sqrt{2}}\left.\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\right|_{\xi=\eta}.$$

4. Functional relationship between solutions of the Cauchy problem for equations (11_α) **and** (11_0)**.** First, we present some properties of the operator L_α, that are necessary for further studies.

1^0. If is a $u_\alpha-$ solution of the equation $L_\alpha u = 0$, then the function

$$u_{2-\alpha} = t^{\alpha-1} u_\alpha \tag{19}$$

is a solution of the equation $L_{2-\alpha} u = 0$.

2^0. If u_α is a solution of the equation $L_\alpha u = 0$, then the function

$$\frac{1}{t}\frac{\partial u_\alpha}{\partial t} = u_{\alpha+2} \tag{20}$$

is a solution of the equation $L_{\alpha+2} u = 0$.

3^0. The operator L_α has the property

$$L_\alpha u_\alpha = t^{1-\alpha} L_{2-\alpha}(t^{\alpha-1} u_\alpha). \tag{21}$$

These properties are established in the same way as they were proved ([Weinstein, 1954]) for the equation

$$\Delta_x u - u_{tt} - \frac{\alpha}{t} u_t = 0. \tag{22}$$

From equality (19) we have $u_{2-\alpha-2p} = t^{\alpha+2p-1} u_{\alpha+2p}$ to which, applying formula (20) p times, and then (19), we obtain

$$u_{2-\alpha} = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^p (t^{\alpha+2p-1} u_{\alpha+2p}). \tag{23}$$

Let $p \geq 0$, $q \geq 0$ be the smallest integers satisfying the inequalities $\alpha+2p \geq m-1$, $2-\alpha+2q \geq m-1$.

Proposition 1. If $u_{0,n}^{k,2}(r,t)$ is a solution to the Cauchy problem for equation (11_0)) satisfying the condition

$$u_{0,n}^{k,2}(r,0) = 0,\ \frac{\partial}{\partial t} u_{0,n}^{k,2}(r,0) = \nu_n^k(r), \tag{24}$$

then function

$$u_{\alpha,n}^{k,2}(r,t) = \gamma_{-\alpha} t^{-\alpha} \int\limits_0^1 u_{0,n}^{k,2}(r,\xi t)\xi(1-\xi^2)^{-\frac{\alpha}{2}-1} d\xi \equiv \gamma_{-\alpha}\Gamma\left(-\frac{\alpha}{2}\right) D_{0t^2}^{\frac{\alpha}{2}} u_{0,n}^{k,2}(r,t), \tag{25}$$

for $\alpha < 0$ it will be a solution of the equation (11_α), satisfying the condition

$$u_{\alpha,n}^{k,2}(r,0) = 0,\ \lim_{t\to 0} t^\alpha \frac{\partial}{\partial t} u_{\alpha,n}^{r,2} = \nu_n^k(r). \tag{26}$$

If $0 < \alpha < 1$, then the function

$$\begin{aligned} u_{\alpha,n}^{k,2}(r,t) = \gamma_{2-k+2q}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q \left[t^{1-k+2q}\int\limits_0^1 u_{0,n}^{k,1}(r,\xi t)(1-\xi^2)^{q-\frac{\alpha}{2}} d\xi\right] \equiv \\ \equiv \gamma_{2-\alpha+2q} 2^{q-1}\Gamma(q_1 - \tfrac{\alpha}{2}+1) D_{0t^2}^{\frac{\alpha}{2}-1}\left[\frac{u_{0,n}^{k,1}(r,t)}{t}\right] \end{aligned} \tag{27}$$

is a solution of the equation (11_α) with the initial data (26), where $\sqrt{\pi}\Gamma(\frac{\alpha}{2})\gamma_\alpha = 2\Gamma(\frac{\alpha+1}{2})$, $\Gamma(z)$ is the gamma function, D_{0t}^α is the Riemann-Liouville operator [Nakhushev, 2006], and $u_{0,n}^{k,1}(r,t)$ is a solution of equation (11_0) with the initial conditions

$$u_{0,n}^{k,1}(r,0) = \frac{\nu_n^k(r)}{(1-\alpha)(3-\alpha)...(2q+1-\alpha)},\ \frac{\partial}{\partial t} u_{0,n}^{k,1}(r,0) = 0. \tag{28'}$$

Proposition 2. If $u_{0,n}^{k,1}(r,t)$ is a solution to the Cauchy problem for equation (11_0) satisfying the condition

$$u_{0,n}^{k,1}(r,0) = \tau_n^k(r),\ \frac{\partial}{\partial t} u_{0,n}^{k,1}(r,0) = 0, \tag{28}$$

then function

$$u_{\alpha,n}^{k,1}(r,t) = \gamma_\alpha \int\limits_0^1 u_{0,n}^{k,1}(r,\xi t)(1-\xi^2)^{\frac{\alpha}{2}-1} d\xi \equiv 2^{-1}\gamma_\alpha\Gamma\left(\frac{\alpha}{2}\right) t^{1-\alpha} D_{0t^2}^{-\frac{\alpha}{2}}\left[\frac{u_{0,n}^{k,1}(r,t)}{t}\right], \tag{29}$$

for $\alpha > 0$ is a solution of equation (11_α), satisfying condition (28).

Proposition 3. If $u_{0,n}^{k,1}(r,t)$ is a solution to the Cauchy problem for equation (11_0) satisfying condition (28), then the function

$$u_{1,n}^{k,1}(r,t) = \int_0^1 u_{0,n}^{k,1}(r,\xi t)(1-\xi^2)^{-\frac{1}{2}} \ln[t(1-\xi^2)]d\xi \tag{30}$$

is a solution to the problem for the equation $L_1 u_{1,n}^k = 0$ with initial data

$$\left.\frac{u_{l,n}^{k,1}}{\ln t}\right|_{t=0} = \tau_n^k(r). \tag{31}$$

The evidence for the above statements is established similarly how they were proved for equation (22) and multidimensional wave equations $\Delta_x u - u_{tt} = 0$ [Aldashev, 1991; Aldashev,1976; Tersenov, 1973; Tersenov, 1982].

We give some corollaries from Propositions 2, 3. We first consider the case $\alpha < 0$, $\alpha \neq -(2r+1)$, $r = 0, 1, \ldots$. If $u_{0,n}^{k,1}(r,t)$ is the solution of the Cauchy problem for (11_0) with data

$$u_{0,n}^{k,1}(r,0) = \frac{\tau_n^k(r)}{(\alpha-1)...(\alpha+2p-1)},\ \frac{\partial}{\partial t}u_{0,n}^{k,1}(r,0) = 0, \tag{32}$$

then it follows from statement 2 that

$$u_{\alpha+2p,n}^{k,1}(r,t) = \gamma_{\alpha+2p}\int_0^1 u_{0,n}^{k,1}(r,\xi t)(1-\xi^2)^{\frac{\alpha}{2}+p-1}d\xi$$

is a solution of the equation $L_{\alpha+2p}u = 0$, satisfying the initial condition(32).

Then from relations (23) and (19) it follows that the function

$$u_{\alpha,n}^{k,1}(r,t) = t^{1-\alpha}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^p \left(t^{\alpha+2p-1}u_{\alpha+2p,n}^{k,1}\right) \equiv \gamma_{k+2p}2^{p-1}\Gamma(\tfrac{\alpha}{2}+p)t^{1-\alpha}D_{0t^2}^{-\frac{\alpha}{2}}\left[\frac{u_{0,n}^{k,1}(r,t)}{t}\right] \tag{33}$$

is a solution to equation (11_α) and satisfies condition (28).

Now let $\alpha = -(2r+1)$. If $u_{0,n}^{k,1}(r,t)$ is a solution to the Cauchy problem for (11_0) with data (28), then it is easy to obtain from (19), (23) and from Proposition 3 that

$$u_{-(2r+1),n}^{k,1}(r,t) = t^{2(r+1)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{r+1}\left[\int_0^1 u_{0,n}^{k,1}(r,\xi t)(1-\xi^2)^{-\frac{1}{2}}\ln(t(1-\xi^2))d\xi\right] \tag{34}$$

is a solution to the Cauchy problem for (11_α), satisfying the condition (28).

Using [Nakhushev, 2000] the relation (34) can be written as

$$u_{-(2r+1),n}^{k,1}(r,t) = \frac{a}{2}t^{2(r+1)}D_{0t^2}^{r+\frac{1}{2}}\left[\frac{u_{0,n}^{k,1}(r,t)}{t}\right],\ a = \frac{1}{2}\Gamma'(1) - \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} - \ln t. \tag{35}$$

5. Proof of the theorem for problem 1. 1) Case $\alpha < 1$. Given formulas (25) and (27), we reduce problem (11_α), (12) to the Dirichlet problem for (11_0) with data

$$u_{\alpha,n}^{k,2}(r,0) = 0,\ u_{\alpha,n}^{k,2}(1,t) = 0,\ u_{\alpha,n}^{k,2}(r,\beta) = 0,\ k = \overline{1,k_n},\ n = 0,1,..., \tag{36}$$

for $\alpha \leq 0$ and to the Poincare problem for equation (11_0), with the condition

$$\frac{\partial}{\partial t}u_{\alpha,n}^{k,2}(r,0) = 0,\ u_{\alpha,n}^{k,2}(1,t) = 0,\ u_{\alpha,n}^{k,2}(r,\beta) = 0,\ k = \overline{1,k_n},\ n = 0,1,..., \tag{37}$$

for $0 < \alpha < 1$.

The following are shown in [9, 10]: 1) If $\gamma \leq -\mu_{s,n}^2$ then problems (11_0), (36) and (11_0), (37) have only zero solutions; 2)For $\gamma > -\mu_{s,n}^2$ problem (11_0), (36) has only a trivial solution if and only if the condition (7) is satisfied; 3)For $\gamma > -\mu_{s,n}^2$ problem (11_0), (37) has only a zero solution if and only if relation (8) holds.

Further, using Statements 1-3, we establish similar results for the problem (11_α), (12).

2) Case $\alpha = 1$. The solution to problem (11_α), (13)will be sought in the form

$$u_{1,n}^{k}(r,t) = u_{1,n}^{k,1}(r,t) + u_{1,n}^{k,2}(r,t), \tag{38}$$

where $u_{1,n}^{k,1}(r,t)$ is solution of the equation (11_1), with data $\left.\frac{u_{1,n}^{k,1}}{\ln t}\right|_{t=0} = 0$, and $u_{1,n}^{k,2}(r,t)$ is solution of the Poincare problem for (11_1) with the condition

$$\frac{\partial}{\partial t} u_{1,n}^{k,2}(r,0) = 0,\ u_{1,n}^{k,2}(1,t) = -u_{1,n}^{k,1}(1,t),\ u_{1,n}^{k,2}(r,\beta) = -u_{1,n}^{k,1}(r,\beta),\ k = \overline{1,k_n},\ n = 0,1,\dots. \tag{39}$$

By virtue of (30), (18) of $u_{1,n}^{k,1}(r,t) \equiv 0$. Further, using formula (29), we reduce problem (11_1), (39) to the Poincare problem (11_0), (37).

Using formulas (21), (19) problem (11_α), (14) is reduced to the case $\alpha < 1$. studied.

Thus, it follows from (10) that Theorem 1 is valid for Problem 1.

6. Proof of Theorem 1 for Problem 2. Now we consider Problem 2, which is reduced to problems (11_α), (15)and (11_α), (16).

If $\alpha \geq 0$, then it follows from (29) that problem (11_α), (15) reduces to the Poincare problem for equation (11_0) with data(37).

For $\alpha < 0$, $\alpha \neq -(2r+1), r = 0,1,\dots$ we will look for a solution to problem (11_α), (16) in the form (38), where $u_{\alpha,n}^{k,2}(r,t)$ is a solution to the Cauchy problem for (11_α) with the condition

$$u_{\alpha,n}^{k,2}(r,0) = 0,\ \lim_{t\to 0} t^\alpha \frac{\partial}{\partial t} u_{\alpha,n}^{k,2}(r,t) = 0, \tag{40}$$

and $u_{\alpha,n}^{k,1}(r,t)$ is solution of the Poincare problem for (11_α) with condition (39).

Problem (11_α), (40) by virtue of formula (25) reduces to the homogeneous Cauchy problem for (11_0) with data $u_{0,n}^{k,2}(r,0) = 0$, $\frac{\partial}{\partial t} u_{0,n}^{k,2}(r,t) = 0$, which has the trivial solution that follows from (18).

Problem (11_α), (39) by virtue of (33) is reduced to Poincare problem (11_0), (37).

Further, let $\alpha = -(2r+1)$. We look for a solution to problem (11_α), (16) in the form (38), where $u_{\alpha,n}^{k,2}(r,t)$ is the solution to the Cauchy problem (11_α), (40), and $u_{\alpha,n}^{k,1}(r,t)$ is solution to the Poincare problem for (11_α) with the condition (39).

Since $u_{\alpha,n}^{k,2}(r,t) \equiv 0$, as shown earlier, by virtue of (35) problem (11_α), (39) reduces to the Poincare problem (11_0), (37).

Therefore, the validity of theorem 1 follows from (10) and it is proved for problem 2.

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