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QUASICLASSICAL APPROXIMATION OF SOLUTIONS OF BOUNDARY CONVECTIVE-TYPE PROBLEMS OF HEAT AND MASS TRANSFER

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Abstract. The development of theoretical methods of analysis of heat and mass transfer processes requires involvement of mathematical physics apparatus first of all. And here there are some problems. In first, heat and mass transfer processes are described by very difficult differential equations. In second, practical difficulties are appeared with definition of parameters and calculation of turbulent heat and mass transfer. In this paper we suggest the approximate method of solving the internal convective heat and mass transfer problems based on the combined use of the Laplace transform and quasiclassical approximation. As an example of the method implementation, the boundary value turbulent heat and mass transfer problem under Dirichlet boundary condition for the motion of medium in a cylindrical channel is considered. The results of numerical analysis of the obtained solutions in smooth and rough channels are presented. The examples of determining the eigenvalues, eigenfunctions and solution constants are given.

Keywords: Boundary Value Problems, Convective Heat and Mass Transfer, Laplace Transform, Quasiclassical Approximation
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оригинальное исследование

КВАЗИКЛАССИЧЕСКОЕ ПРИБЛИЖЕНИЕ РЕШЕНИЙ КРАЕВЫХ ЗАДАЧ КОНВЕКТИВНОГО ТЕПЛО- И МАССОПЕРЕНОСА

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Аннотация. Предложен приближенный метод решения внутренних задач конвективного тепло- и массопереноса, основанный на совместном применении интегрального преобразования Лапласа и квазиклассического приближения. В качестве примера реализации метода рассмотрено решение краевой задачи турбулентного тепло- и массопереноса для случая движения среды в цилиндрическом канале при граничном условии первого рода. Приведены результаты численного анализа полученных решений для гладких и шероховатых каналов. Даны примеры определения собственных значений, собственных функций и постоянных решений краевой задачи.

Ключевые слова: краевые задачи, конвективный тепло- и массоперенос, преобразование Лапласа, квазиклассическое приближение

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1. Introduction. The processes of heat and mass transfer in turbulent motion are interconnected. At the same time, the main transfer equations have an identical basis since the differential equations, which are used to describe these phenomena, relate to the general diffusion equation [7, 8].

At present, the processes of heat and mass transfer and hydrodynamics are rather fully studied in turbulent motion of the medium in smooth and rough channels. Turbulent flow characteristics were investigated by Reynolds, Prandtl, Launder, Ryhardtom, Deysler, Kolmogorov, Spalding, Kutateladze and many other authors [9, 13, 18, 22, 24, 25, 27]. Analytical methods of solving of heat and mass transfer problems are quite diverse [7, 8, 9, 12, 13, 14, 15, 17, 25]. However, they relate to the narrow class of simple tasks.

Recently, much attention in the problem of turbulent heat and mass transfer in the channels has been paid to researching the transfer phenomena and improving models of turbulence in the boundary layer [11, 12, 19, 23, 25, 26], as well as to the numerical solving the boundary problems of convective heat and mass transfer [4, 5, 25, 26]. The absence of widely accepted models in many cases and the application of many different models of turbulence require effective methods of analytical solving the boundary problems, when the functions characterizing turbulent transfer are defined in general form.

In turn, modern numerical methods allow to solve many boundary value problems of heat and mass transfer. But at the same time, there is a certain loss of presentation and universality of solutions is observed. Also, there is a very important requirement for the researcher to have sufficient experience and practice with complicated numerical models.

For this reason, the specialists are interested in use of approximate analytical methods of solving convective heat and mass transfer problems [2, 10, 21, 25]. These methods have a number of advantages, because they allow to effectively construct the solutions of many boundary problems of heat and mass transfer. In this case, various modifications of the approximate Galerkin method [2, 21] are fairly used. They allow to obtain solutions that are presented in finite series using coordinate functions.

The disadvantages of this method are the significant increase of analytical and computational complexity with an increase in number of terms of the series in the approximate solution and also the problem of choosing the satisfactory coordinate functions. To our mind, there is insufficient attention has been paid to the approximate analytical methods of solving the boundary problems of heat and mass transfer, which allow to present general solutions in the form of infinite series, and to the methods of justification of coordinate functions, which correspond to the particularity of the solved problem.

The purpose of this work is to present an analytical method for solving boundary value problems of heat and mass transfer based on the Laplace transform and asymptotic decomposition of the images with the subsequent constructing of the general approximate solution. This method allows to receive solutions of rather wide class of heat and mass transfer problems in an approximate analytical form.

2. Approximate analytical method for solving boundary value problems of heat and mass transfer. Approximate analytical method for solving boundary value problems of heat and mass transfer. The boundary value problem of a stationary convective heat and mass transfer at the motion of various particles in the channels of any geometrical form leads to the integration of differential equation, which can be presented in the form:

$$\overline{W}(M)\frac{\partial\theta}{\partial z} = a\,div[\overline{f}(M)\;grad\;\theta(M,z)] + V_*(M,z), \tag{1}$$

under boundary conditions

$$\ell_s[\theta(M,z)]_{\tilde{s}} = \varphi(M_s,z), \tag{2}$$

and initial conditions

$$[\theta(M,z)]_{z=0} = \varphi_0(M), \tag{3}$$

where $\tilde{s}$ is the contour of surface Ω; Ω is the surface of the cross-section removed by the distance z from the entrance of the channel; $M = M(x,y)$ is the current point of the surface Ω ($M \in \Omega$); M_s is the point on the contour $\tilde{s}$; ℓ_s is the linear differentiation operator defined on the contour $\tilde{s}$; $\ell_s[\theta(M,z)]_{\tilde{s}} = \alpha\theta + \beta\frac{\partial\theta}{\partial\vec{n}}|_{\tilde{s}}$; $\theta(x,y,z)$ is the heat or mass transfer potential; $\overline{W}(M)$, $\overline{f}(M)$ are the velocity of the motion of the medium and parameters of turbulent transfer in the cross-section z of the channel; a is the coefficient of thermal diffusivity (diffusion) of the medium; $V_*(M,z)$ is the function describing sources or drains; α, β are the constants.

The condition (2) can be represented in the form of Dirichlet or Neumann boundary conditions. In the first case, the zero-order differentiation operator has a form:

$$\ell_s[\theta]_{\tilde{s}} = [\theta(M,z)]_{\tilde{s}},$$

In the second case, the differentiation operator of the fist order can be represented as: $\ell_s[\theta]_{\tilde{s}} = \left[\lambda(M)\frac{\partial\theta}{\partial\vec{n}}\right]_{\tilde{s}}$, where $\lambda(M)$ is the function characterizing heat or mass transfer on the channel wall; $\vec{n}$ is the external normal to a contour $\tilde{s}$.

After applying the Laplace transform on coordinate z the boundary value problem (1) – (3) is reduced to:

$$L[\overline{\theta}(M,p)] + \frac{1}{a}\overline{R}(M,p) = 0; \tag{4}$$

$$\ell_s[\overline{\theta}(M,p)]_{\tilde{s}} = \overline{\varphi}(M_s,p), \tag{5}$$

where $\overline{\theta}(M,p) = \int_0^\infty \theta(M,z)\exp(-pz)dz$ is the Laplace transform of the function $\theta(M,z)$ on coordinate z; p is the Laplace transform parameter; $L[\overline{\theta}(M,p)] = div[\overline{f}(M)\ grad\overline{\theta}] - p\overline{W}(M)\overline{\theta}$ is the differentiation operator; $\overline{V_*}(M,p) = \int_0^\infty V_*(M,z)\exp(-pz)dz$; $\overline{\varphi}(M_s,p) = \int_0^\infty \varphi(M_s,z)\exp(-pz)dz$; $\overline{R}(M,p) = \overline{V_*}(M,p) + \overline{W}(M)\varphi_0(M)$.

In most cases, it is rather difficult or simply impossible to obtain an exact solution of the problem (4)–(5). Therefore, development of approximate analytical methods for solving such problems is of considerable interest. The approximate method [21] is based on combined use of the Laplace transform and Ritz method or the orthogonal Bubnov-Galerkin method. It has great opportunities for solving the number of transfer problems. This method allows to reduce a study of heat and mass transfer problems to solving the algebraic systems and to receive approximate analytical dependences. However, at constructing the solutions there is a problem to choose the optimal system of coordinate functions on which the solution depends. In particular, at solving the boundary value problems of convective heat and mass exchange in turbulent flows the correct accounting of the boundary layer is of great importance. In this case, it is almost impossible to select the optimal system of coordinate functions in advance. Additional difficulties, when using Ritz or Bubnov-Galerkin methods, are associated with the fact that with an increase in number of coordinate functions up to three or more the volume of computational work increases dramatically. Therefore, the method [21] did not gain wide distribution in solving the transfer problems for turbulent flows.

We propose a new method of an approximate solution of boundary value problems. It is based on the preliminary analysis of the solution in the Laplace transform domain and asymptotic decomposition of the image for a great value of transform parameter [3, 14]. After transition to the original domain the distribution of potentials for an initial stage of heat and mass transfer process is determined. For the preliminary analysis of the solution in the Laplace transform domain the quasiclassical approximation is used [20, 16]. This method allows to obtain the solutions of wide class of heat and mass transfer problems in an approximate analytical form.

Consider a technique of method's application. Suppose that in the Laplace transform domain the approximate solution of the problem (4)–(5) by means of a quasiclassical approximation method is obtained at $p \to \infty$. By asymptotically expanding the obtained solution into rapidly converging series for the great values p and the subsequent transition to the original domain the distribution of potential for the initial stage of heat or mass transfer process can be found by the Lykov method for small values z [14].

If the asymptotic approximation $\overline{\Phi}(M,p)$ of the function $\overline{\theta}(M,p)$ at $p \to \infty$ and the corresponding original $\Phi(M,z)$ of $\theta(M,z)$ at $z \to 0$ are found, then it is possible to compose the residual of the equation (1) in the original domain.

There is a general solution of the mixed problem (1)–(2) for the equation of parabolic type, which, accoording to [7], in some cases at $\varphi = 0$ and $V_* = 0$ can be presented by Fourier expansion on eigenfunctions χ_n of operator $L[\theta]$ as:

$$\theta(M,z) = \sum_{n=0}^{\infty} A_n\chi_n(M)\exp(-p_n z),$$

where p_n are the eigenvalues; A_n are the constants. These values can be determined from the condition of equality of functions $\Phi(M,z)$ and $\theta(M,z)$ for $z = 0$ and small values z:

$$\theta(M,z)|_{z=0} = \Phi(M,z)|_{z=0} = \varphi_0(M); \tag{6}$$

$$\lim_{z\to\delta_*}[\theta(M,z) - \Phi(M,z)] = 0, \tag{7}$$

where δ_* is a value close to zero.

Let coordinate functions $\psi_n(M)$, which are approximated eigenfunctions $\chi_n(M)$, be found and satisfy boundary conditions (2). When applying one of residual's minimization methods, one can obtain a system of the linear algebraic equations for finding of the constants A_n and a system of linear or nonlinear algebraic equations for finding of the eigenvalues. The best approximation of the coordinate functions $\psi_n(M)$ is determined from the asymptotic solution $\Phi(M,z)$. Let us construct an approximate solution of the equation (1) in a form:

$$\theta_n(M,z) = \sum_{n=0}^{N} A_n\Psi_n(M)\exp(-p_n z), \tag{8}$$

and we will reduce the process of solving the boundary value problem (1)–(3) to aproximation of function $\Phi(M,z)$ by the series (8) for small values z.

To determine the coefficients A_n on the basis of equation (6), we construct the residual:

$$\varepsilon(M,A_n) = \left[\varphi_0(M) - \sum_{n=0}^{N} A_n\Psi_n(M)\right]^2$$

and select the coefficients A_n from the condition of minimum $\varepsilon(M, A_n)$. Thus, we will receive the algebraic system of equations on coefficients A_n:

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial\varepsilon}{\partial A_1} = \iint_{\Omega}\left[\varphi_0(M) - \sum_{n=0}^{N} A_n\Psi_n(M)\right]\Psi_1(M)dw = 0; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots \\ \dfrac{1}{2}\dfrac{\partial\varepsilon}{\partial A_n} = \iint_{\Omega}\left[\varphi_0(M) - \sum_{n=0}^{N} A_n\Psi_n(M)\right]\Psi_n(M)dw = 0, \end{cases}$$

whence the coefficients A_n are determined by known methods.

If $\Psi_n(M)$ is taken to be an orthogonal system of coordinate functions in the domain Ω with respect to a weight function $\rho(M)$, then the coefficients A_n are determined from the equations:

$$A_n = \frac{\iint_{\Omega}\rho(M)\varphi_0(M)\Psi_n(M)dw}{\iint_{\Omega}\rho(M)\Psi_n^2(M)dw}.$$

To determine the eigenvalues p_n from the equation (7), we take the residual in a form:

$$\varepsilon(p_n, M, z) = \left[\Phi(M,z) - \sum_{n=0}^{N} A_n\Psi_n(M)\exp(-p_n z)\right]^2$$

and choose the eigenvalues p_n in such a way that the residual has its smallest value at $z \to 0$. By expanding exponents into a series and limiting to two terms of a series, we obtain the algebraic system of equations:

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial\varepsilon}{\partial p_1} = \int_0^{\delta_*}\iint_{\Omega}\left[\Phi(M,z) - \sum_{n=0}^{N}\left[A_n\Psi_n(M) - A_n\Psi_n(M)p_n z\right]\right]\times A_1\Psi_1(M)zdwdz = 0; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots \\ \dfrac{1}{2}\dfrac{\partial\varepsilon}{\partial p_n} = \int_0^{\delta_*}\iint_{\Omega}\left[\Phi(M,z) - \sum_{n=0}^{N}\left[A_n\Psi_n(M) - A_n\Psi_n(M)p_n z\right]\right]\times A_n\Psi_n(M)zdwdz = 0; \end{cases}$$

If we take $\Psi_n(M)$ as an orthogonal system of coordinate functions in a domain Ω with respect to a weight function $\rho(M)$, then the eigenvalues p_n are determined from the equations:

$$p_n = \frac{\int_0^{\delta_*}\iint_{\Omega}\left[A_n\Psi_n(M) - \Phi(M,z)\right]\rho(M)\Psi_n(M)dwdz}{\int_0^{\delta_*}\iint_{\Omega}A_n\rho(M)\varphi_n^2(M)zdwdz}.$$

Thus, the search procedure for an approximate solution of the boundary value problem (1)—(3) consists of the following stages:
1) using the Laplace transform, the initial convective heat and mass transfer problem is reduced to the boundary value problem for the differential equation of the second order;
2) using the method of quasiclassical approximation, we determine the asymptotic solution $\overline{\Phi}(M,p)$ of the differential equation in the Laplace transform domain for large values p;
3) by analyzing the image $\overline{\Phi}(M,p)$ for great values of parameter p, we receive the original $\Phi(M,z)$ for small values z;
4) based on the form of the function $\Phi(M,z)$, the basic coordinate functions $\psi_n(M)$ are determined. Thus, we obtain an asymptotic approximation of the corresponding eigenfunc- tions $\chi_n(M)$ in the space, in which the approximate solution of the boundary value problem is found;
5) the approximate solution of the problem is complicated in the form (8);
6) approximation of the function $\Phi(M,z)$ by functions $\theta_n(M,z)$, using residual's minimization in the domain Ω, is found for small values z;
7) the system of algebraic equations for determining of the constants A_n and the eigenvalues p_n is being formed and solved.

The proposed method of obtaining approximate solutions can be used to solve the problems of non-stationary heat conductivity and diffusion, convective heat and mass exchange in pipes and channels under various boundary conditions and also the problems of heat and mass transfer in various bodies at variable coefficients of heat conductivity and diffusion. The method allows to receive approximate analytical solutions of boundary value problems in a simple way that are currently being solved numerically. The received solutions satisfy the boundary conditions and asymptotically coincide with the exact solution for an initial stage of heat and mass exchange. An advantage of the method is the possibility of determining the asymptotic approximations of coordinate functions (basis) and constructing a residual for determining the constants and eigenvalues in the original domain. Apart from that, we can obtain the approximate solution in the form of infinite series on the obtained approximations of the basis. The disadvantage of this method is the possibility of an error in the approximation compared to the exact solution for the last stages of the heat and mass exchange process (at great values z).

3. Scheme for receiving a quasiclassical approximation of solutions of internal convective heat and mass transfer problems. Consider the internal problem of convective heat and mass transfer, when a homogeneous liquid or gas enters a fairly long channel of rectangular or circular cross-section.

At the initial region of the channel ($z \le 0$) the steady flow is formed in such way that the velocity field at the entrance of the liquid into the channel ($z \ge 0$) becomes stationary. The boundary condition of heat and mass exchange with the external environment on the wall of a pipe is specified. This condition is described by homogeneous Dirichlet and Neumann boundary conditions. The internal problem of convective heat and mass transfer is reduced to determining the tranfer potentials θ.

Distribution of potentials of heat and mass transfer for the steady liquid flow in the flat channel ($n = 0$) or cylindrical channel ($n = 1$) satisfies the following equation:

$$Pe\ w(\xi)\xi^n \frac{\partial\theta}{\partial\eta} = \frac{\partial}{\partial\xi}\left((1 + f(\xi))\xi^n \frac{\partial\theta}{\partial\xi}\right), 0 \le \xi \le 1, 0 \le \eta \le \infty. \qquad (9)$$

Under boundary conditions

$$\theta(\xi,\eta) = 1,\ \eta = 0 \qquad (10)$$

$$\frac{\partial\theta}{\partial\xi} = 0,\ \xi = 0 \qquad (11)$$

where $f(\xi)$ is the function, which is characterizing the parameters of turbulent heat and mass transfer, having continuous derivatives of the second order and $f(\xi) \ge 0, 0 \le \xi \le 1$. Here $\xi = y/y_0$, $\eta = z/z_0$ are dimensionless spatial coordinates, Pe is the heat or diffusion Peclet number.

In addition to the conditions (10)–(11), for solving the equation (9) it will be necessary to set a boundary condition on the channel wall. Under Dirichlet boundary conditions for $\xi = 1$ we have $\theta(\xi,\eta) = 0$, and under Neumann boundary conditions at $\xi = 1$: $\dfrac{\partial\theta(\xi,\eta)}{\partial\xi} = 1$.

Using integral Laplace transfom for the equation (9) on the variable η, we obtain equation:

$$\frac{d}{d\xi}\left(k(\xi)\frac{d\overline{\theta}}{d\xi}\right) - p\cdot Pe\cdot r(\xi)\overline{\theta} = 0, \qquad (12)$$

where $k(\xi)$ and $r(\xi)$ are anywhere positive functions, having continuous derivatives of the second order in the range $0 \le \xi \le 1$.

We construct a quasiclassical approximation of solutions of the equation (12) at $p \to \infty$ [16, 20]. By replacing $\overline{\theta}(\xi) = \varphi(\xi)U(S)$, $S = S(\xi)$, we lead equation (12) to a form:

$$U''(S) - [p + g(S)]\, U(S) = 0, \qquad (13)$$

where $S(\xi) = \int_{\xi_0}^{\xi}\sqrt{\frac{r(t)}{k(t)}}dt$, $\varphi(\xi) = [k(\xi)r(\xi)]^{-1/4}$, $0 < \xi_0 < 1$.

It is known [20, 16], that the solutions of the equation (13) at $p \to \infty$ asymptotically approximate to the solutions of the equation $U''(S) - pU = 0$, which, in turn, have a form: $U(S) = A\exp(\mu S) + B\exp(-\mu S)$, where $\mu = \sqrt{p}$, A, B are the constants. Returning to the previous variables in a given equation, we receive that the solutions of the equation (12) for $p \to \infty$ is as follows:

$$\overline{\theta}(\xi) = \frac{1}{\sqrt{k(\xi)r(\xi)}}[A\exp(\varsigma(\xi)) + B\exp(-\varsigma(\xi))], \rho(\xi) = \sqrt{p\left|\frac{r(\xi)}{k(\xi)}\right|}, \varsigma(\xi) = \int_{\xi_0}^{\xi}\rho(t)dt. \qquad (14)$$

Also, the solving the equation (12) is of practical interest at $p \to \infty$, when functions $k(\xi)$ and $r(\xi)$ have the expressions: $k(\xi) = \xi^m\overline{k}(\xi)$; $r(\xi) = \xi^n\overline{r}(\xi)$, and functions $\overline{k}(\xi)$ and $\overline{r}(\xi)$ are strictly positive in the range $0 \le \xi \le 1$.

As shown in [16], quasiclassical approximation of solutions of the equation (12) in this case can be presented in the form:

$$\overline{\theta}(\xi) = \sqrt{\frac{\varsigma(\xi)}{k(\xi)\rho(\xi)}}[AI_\nu(\varsigma(\xi)) + BK_\nu(\varsigma(\xi))],$$

where $\rho(\xi) = \sqrt{p\left|\frac{r(\xi)}{k(\xi)}\right|}$, $\varsigma(\xi) = \int_0^{\xi}\rho(t)dt$, $\nu = \frac{|m-1|}{n-m+2}$, $I_\nu(z)$, $K_\nu(z)$ are the modified Bessel functions, $n - m + 2 > 0$.

Taking into account the corresponding boundary conditions on cross coordinate, it is possible to receive asymptotic approximations of solutions of the equation (12) in the Laplace transform domain for great values p. If it is possible to determine the original of the corresponding image, the approximate solution of this equation can be obtained by the method described in the previous section.

4. Heat and mass transfer in a cylindrical channel under Dirichlet boundary conditions and variable flow rate. We define the solution of the boundary value problem (9)–(11) under Dirichlet boundary conditions in the case of variable flow rate. Applying the Laplace transform to the system of equations on the longitudinal coordinate η, we obtain:

$$\frac{d}{d\xi}\left((1+f(\xi))\xi\frac{d\overline{\theta}}{d\xi}\right)-Pe\cdot p\cdot\xi\cdot W(\xi)\left(\overline{\theta}-\frac{1}{p}\right)=0,\ (\overline{\theta})_{\xi=1}=0;\left(\frac{\partial\overline{\theta}}{\partial\xi}\right)_{\xi=0}=0. \tag{15}$$

Let us determine the solution of the equation (15) for the main area of a turbulent flow and for very thin parietal section. In the first case, the solution of the equation (15), taking into account (14) and boundary condition for $\xi=0$, can be approximately presented at $p\to\infty$, $0\le\xi\le 1-\delta$, in the form:

$$\overline{\theta}_{+}-\frac{1}{p}=c_1\frac{F^{1/2}(\xi)I_0(F(\xi)\sqrt{Pe\cdot p})}{\sqrt{\xi}[(1+f(\xi))W(\xi)]^{1/4}}, \tag{16}$$

δ is the thickness of the boundary layer near the wall; c_1 is a constant, $F(\xi)=\int_0^{\xi}\left(\frac{W(t)}{1+f(t)}\right)^{1/2}dt$.

In this case, potential of heat and mass transfer will be described by two functions: $\theta_{+}(\xi,\eta)$ – in the main area of turbulent flow for $0\le\xi\le 1-\delta$ and $\theta_{-}(\xi,\eta)$ – in thin parietal section for $1-\delta<\xi\le 1$. This is explained by the fact that the equation (15) has a singularity at $\xi=1$. In this regard, the solution of (15) cannot be extended to the boundary layer zone for $1-\delta\le\xi\le 1$. In the boundary layer $f(\xi)=0$ and the velocity $W(\xi)$ and thickness of the boundary layer are equal to $W(\xi)=\dfrac{\lambda_* Re}{16}(1-\xi);\delta=5\left(Re\sqrt{\frac{\lambda_*}{32}}\right)^{-1}$. According to this, the equation (15) for the boundary layer at $1-\delta\le\xi\le 1$ has an approximate form:

$$\frac{d^2\overline{\theta}_{-}}{d\xi^2}-Pe\cdot p\cdot r_*(1-\xi)\left(\overline{\theta}_{-}-\frac{1}{p}\right)=0;r_*=\frac{\lambda_* Re}{16}.$$

The solution of this equation is as follows [11]:

$$\overline{\theta}_{-}(\xi,p)-\frac{1}{p}=\sqrt{1-\xi}c_1I_{1/3}\left(\frac{2}{3}\sqrt{Pe\cdot p\cdot r_*}(1-\xi)^{3/2}\right)+$$

$$+\sqrt{1-\xi}c_2K_{1/3}\left(\frac{2}{3}\sqrt{Pe\cdot p\cdot r_*}(1-\xi)^{3/2}\right).$$

For the thin boundary layer at $\xi\to 1$ we replace the functions $I_{1/3}(z)$ and $K_{1/3}(z)$ for small values z by their limits [16, 1]: $I_{1/3}(z)\approx\frac{1}{\Gamma(4/3)}\left(\frac{z}{2}\right)^{1/3};K_{1/3}(z)\approx\frac{1}{2}\Gamma(1/3)\left(\frac{z}{2}\right)^{-1/3}$. Therefore, the solution of (15) in a very thin boundary layer can be approximated by linear function: $\overline{\theta}_{-}(\xi,p)-\frac{1}{p}=c_2(1-\xi)+const$. Considering a boundary condition on the wall, we receive:

$$\overline{\theta}_{-}(\xi,p)=c_2(1-\xi). \tag{17}$$

Coefficients c_1 and c_2 from (16) and (17) can be found from the conjugation condition of the functions $\overline{\theta}_{+}(\xi,p)$ and $\overline{\theta}_{-}(\xi,p)$ on a border of the boundary layer for $\xi=1-\delta$: $\overline{\theta}_{+}=\overline{\theta}_{-}$; $\frac{\partial\theta_{+}}{\partial\xi}=\frac{\partial\theta_{-}}{\partial\xi}$.

Determining the constants c_1 and c_2 from these conditions and passing into the original domain, we will obtain the solutions $\overline{\theta}_{+}(\xi,\eta)$ and $\overline{\theta}_{-}(\xi,\eta)$ for small values η:

$$\overline{\theta}_{+}(\xi,\eta)=1-\frac{F^{1/2}(\xi)W^{-1/4}(\xi)}{\sqrt{\xi}(1+f(\xi))^{1/4}}\left(1-\sum_{n=1}^{\infty}\tilde{A}_nJ_0\left(F(\xi)\frac{\mu_n}{m}\right)\exp\left(-\frac{\mu_n^2\eta}{m^2Pe}\right)\right);$$

$$\overline{\theta}_{-}(\xi,\eta)=\frac{1-\xi}{\delta}-a_1n\frac{1-\xi}{\delta}\left(1-\sum_{n=1}^{\infty}\tilde{A}_nJ_0(\mu_n)\exp\left(-\frac{\mu_n^2\eta}{m^2Pe}\right)\right),$$

where

$$\tilde{A}_n=\frac{2\beta}{J_0(\mu_n)(\mu_n^2+\beta^2)};\beta=\frac{a_1+b_1\delta}{b_2\delta}m;n=\frac{1}{a_1+b_1\delta};a_1=\left(\frac{F^{1/2}(\xi)}{\sqrt{\xi}[W(\xi)(1+f(\xi))]^{1/4}}\right)_{\xi=1-\delta};$$

$$b_2 = \left(\frac{F^{1/2}(\xi)W^{1/4}(\xi)}{\sqrt{\xi}[(1+f(\xi))]^{3/4}} \right)_{\xi=1-\delta}; m = \int_0^{1-\delta} \sqrt{\frac{W(\xi)}{1+f(\xi)}} d\xi; b_1 = \left(\frac{F^{1/2}(\xi)}{\sqrt{\xi}[W(\xi)(1+f(\xi))]^{1/4}} \right)_{\xi=1-\delta};$$

μ_n are the roots of the equation $\beta J_0(\mu) = \mu J_1(\mu)$; $f(\xi) = 0$ at $1-\delta \le \xi \le 1$.

Determine the general solution of the equation (9) in the range $0 \le \xi \le 1$ in a form:

$$\overline{\theta}_+(\xi,\eta) = \sum_{n=0}^{\infty} A_n \Psi_n(\xi) \exp\left(-p_n^2 \frac{\eta}{Pe}\right). \tag{18}$$

Denote by $\Psi_n(\xi) = \frac{F^{1/2}(\xi)}{\sqrt{\xi}[(1+f(\xi))W(\xi)]^{1/4}} J_0\left(F(\xi)\frac{\mu_n}{m}\right)$ the eigenfunctions. These functions satisfy boundary conditions. They are asymptotic approximations of eigenfunctions for $\eta \to 0$ in the range $1-\delta \ge \xi \ge 0$. Functions $\Psi_n(\xi)$ are orthogonal in the range $0 \le \xi \le 1-\delta$ with the weight $\xi W(\xi)$ as:

$$\int_{-1+\delta}^{1-\delta} \xi W(\xi)\Psi_n(\xi)\Psi_k(\xi) d\xi = \begin{cases} 0, n \ne k; \\ m + \frac{m}{2\mu_n} sin(2\mu_n), n = k. \end{cases}$$

In the equation (18) the coefficients A_n and eigenvalues p_n are determined by the previously specified method and are equal to

$$A_n = \frac{2R_n}{m^2(J_0^2(\mu_n) + J_1^2(\mu_n))};$$

$$p_n^2 = \frac{nJ_1(\mu_n)\mu_n}{R_n}, R_n = \int_0^{1-\delta} \frac{\sqrt{F(\xi)\xi}W^{3/4}(\xi)}{(1+f(\xi))^{1/4}} J_0\left(F(\xi)\frac{\mu_n}{m}\right) d\xi.$$

In the range $1-\delta \le \xi \le 1$ the solution of the equation (15) is:

$$\overline{\theta}_-(\xi,\eta) = \frac{1-\delta}{\delta} \sum_{n=0}^{\infty} A_n \frac{\sqrt{m}J_0(\mu_n)}{[W(1-\delta)(1+f(1-\delta))]^{1/4}} \exp\left(-p_n^2 \frac{\eta}{Pe}\right). \tag{19}$$

The approximate solutions (18) and (19) of the equation (9) satisfy boundary conditions and tend to zero for great values η, while the functions $f(\xi)$ and $W(\xi)$ are given in general form. The dependence of the potential value θ_m on the average cross-section of the channel on η is as follows:

$$\theta_m(\eta) = 2\int_0^1 fW(\xi)\theta(\xi,\eta) d\xi = 2\sum_{n=0}^{\infty} A_n R_n \exp\left(-p_n^2 \frac{\eta}{Pe}\right).$$

The influence of the change of the value $\theta(\xi,\eta)$ in the boundary layer of thickness δ can be neglected.

Local Nusselt number $Nu(\eta)$ is equal to:

$$Nu(\eta) = -\frac{2}{\theta_m(\eta)}\left(\frac{\partial\theta_-}{\partial\xi}\right)_{\xi=1} = \frac{\sum_{n=0}^{\infty} A_n \sqrt{m} J_0(\mu_n)\exp\left(-p_n^2\frac{\eta}{Pe}\right)}{\delta[W(1-\delta)]^{1/4}\sum_{n=0}^{\infty} A_n R_n \exp\left(-p_n^2 \frac{\eta}{Pe}\right)}.$$

In the cross-sections of a cylindrical pipe, which are remoted from its entry, the Nusselt number is close to the following constant value $Nu_\infty = \frac{\sqrt{m}J_0(\mu_0)}{\delta[W(1-\delta)]^{1/4}R_0} = p_o^2$. In that special case, when the flow velocity in the channel is constant, the solution of the boundary problem (9) has a form:

$$\theta(\xi,\eta) = \sum_{n=0}^{\infty} A_n \Psi_n(\xi) \exp\left(-p_n^2 \frac{\eta}{Pe}\right),$$

where

$$\Psi_n(\xi) = \frac{F^{1/2}(\xi)}{\sqrt{\xi}\,(1+f(\xi))^{1/4}} J_0\left(\frac{\mu_n}{m}F(\xi)\right); F(\xi) = \int_0^1 ((1+f(t))^{-1/2} dt,$$

$$p_n^2 = \frac{J_1(\mu)\mu_n\,(1+f(\xi))^{1/4}}{R_n\sqrt{m}}; A_n = \frac{2R_n}{m^2 J_1^2(\mu_n)}; R_n = \int_0^1 \frac{\sqrt{\xi F(\xi)}}{(1+f(\xi))^{1/4}} J_0\left(\frac{\mu_n}{m}F(\xi)\right) d\xi,$$

μ_n are the roots of Bessel function of the first kind $J_0(\mu)$; $m = F(1)$.

And the Nusselt number is equal to $Nu_\infty = \frac{\mu_0 J_1(\mu_0)}{\sqrt{m}R_0} = p_o^2$.

5. Heat and mass transfer in smooth and rough channels. 5.1. Smooth channels. The numerical analysis of the received solutions was carried out for different types of functions $f(\xi)$ and $w(\xi)$ for the motion of gas medium in flat and cylindrical channels. In the calculations the Prandtl number or Schmidt number varied

Table 1. Smooth channel. The eigenvalues and constants of solutions of heat and mass transfer problem under Dirichlet boundary condition, the constant flow rate $w(\xi) = 1$ and function $f(\xi)$ specified by Martinelli's equations

Таблица 1. Гладкий канал. Собственные значения и постоянные решения задачи о тепло- и массопереносе при граничном условии первого рода, постоянной скорости потока $w(\xi) = 1$ и задании функции $f(\xi)$ уравнениями Мартинелли

Order	Pr or Sc	Re	m	R_n	A_n	Eigenvalues p_n^2
0	0.7	10000	0.3093	0.07187	5.5750	31.24
1				-0.02144	-3.8720	157.20
2				0.00264	0.7479	1602.00
3				-0.00270	-1.0450	1825.00
4				0.00112	0.5491	4949.00
5				-0.00141	-0.8347	4335.00
0	0.7	50000	0.1594	0.03651	10.6600	85.65
1				-0.01211	-8.2330	388.40
2				0.00159	1.7020	3692.00
3				-0.00159	-2.3160	4317.00
4				0.00057	1.0430	13660.00
5				-0.00056	-1.2540	15130.00
0	0.7	100000	0.1191	0.02714	14.1900	133.30
1				-0.00925	-11.2550	588.40
2				0.00122	2.3410	5560.00
3				-0.00124	-3.2220	6426.00
4				0.00046	1.1540	19490.00
5				-0.00045	-1.8080	21730.00

from 0.1 to 1.6, and the Reynolds number – from $5 \cdot 10^3$ to 10^6, and the Prandtl and Schmidt turbulent numbers were assumed to be 1.0.

In the process of calculations the function $f(\xi)$, characterizing turbulent thermal conductivity or diffusion, was given as: $\frac{\lambda_T}{\lambda} = \frac{Pr}{Pr_T} \cdot \frac{\nu_T}{\nu}$, $\frac{D_T}{D} = \frac{Sc}{Sc_T} \cdot \frac{\nu_T}{\nu}$, where Pr_T and Sc_T are the Prandtl and Schmidt turbulent numbers. The turbulent kinematic viscosity $\nu_T(\xi)$ for smooth channels was set in the form of Martinelli's equations [22] (the three-layer scheme of a turbulent flow):
a) for laminar layer: $0 \le y^+ < 5; u^+ = y^+; f(\xi) = 0$;
b) for buffer (intermediate) layer: $5 \le y^+ < 30; u^+ = -3.05 + 5.00 \ln y^+; f(\xi) = 0.2y^+ - 1$;
c) for turbulent kernel (logarithm layer): $y^+ \ge 30; u^+ = 5.5 + 2.5 \ln y^+; f(\xi) = 0.4y^+\xi$,
where $y^+ = Re\sqrt{\frac{\lambda_*}{32}}\,(1-\xi)\,; u^+ = w(\xi)\sqrt{\frac{8}{\lambda_*}}$;
Raykhardt's equations (two-layer flow scheme): $f(\xi) = 0.4\left(y^+ - 11th\left(\frac{y^+}{11}\right)\right); 0 \le y^+ \le 50$;
$f(\xi) = 0.133y^+(0.5 + \xi^2)(1 + \xi); 50 < y^+ \le y_0^+$, where $y_0^+ = Re\sqrt{\frac{\lambda_*}{32}}$ is the dimensionless pipe radius;
and Spalding's equations (single-layer scheme):
$y^+ = u^+ + \frac{1}{E}\left[\exp(ku^+) - 1 - ku^+ - \frac{(ku^+)^2}{2!} - \frac{(ku^+)^3}{3!} - \frac{(ku^+)^4}{4!}\right]$;
$f(\xi) = \frac{k}{E}\left[\exp(ku^+) - 1 - ku^+ - \frac{(ku^+)^2}{2!} - \frac{(ku^+)^3}{3!}\right]$, where $k = 0.407$, $E = 10$.

The function $w(\xi)$, characterizing the flow velocity in the channel, was taken in the form of the equations of rod profile and the equations of logarithm profile of velocity. As an example, the eigenvalues and constants of solutions of the boundary value problem of heat and mass transfer for cylindrical channel at constant flow rate in the cross-section are given in Table 1. The made analysis showed that the type of function $f(\xi)$ affects the eigenvalues and constants of the received solutions. Numerical results coincide well with the available data for the eigenvalues and constants of solutions of heat transfer problem in turbulent flow of the medium at the thermally initial site [6]. Heat exchange problems in turbulent flow in a cylindrical channel in this case are solved by separating of variables and numerically determining the eigenvalues and eigenfunctions.

5.2. Rough channels. For this case, as well as for smooth channels, the numerical analysis was carried out at different types of functions $f(\xi)$ and $w(\xi)$, characteristic for the describing of the motion of the medium in a cylindrical channel. At calculating the Prandtl and Schmidt numbers varied from 0.1 to 1.6, the Reynolds number – from $5 \cdot 10^3$ to 10^6, and the Prandtl and Schmidt turbulent numbers were taken to be 1.0. The roughness value varied within $\Delta_s = R_0/k_s = 15 - 500$ (k_s is the roughness height), and the analysis was carried out mainly for homogeneous sand roughness as the most studied. In calculations the function, characterizing turbulent thermal

Table 2. Rough channel. The eigenvalues and constants of solutions of heat and mass transfer problem under Dirichlet boundary condition, the constant flow rate $w(\xi) = 1$ and function $f(\xi)$ specified by Roth's equations, $R_0/k_s = 15$

Таблица 2. Шероховатый канал. Собственные значения и постоянные решения задачи о тепло- и массопереносе при граничном условии первого рода, постоянной скорости потока $w(\xi) = 1$ и задании функции $f(\xi)$ уравнениями Ротта, $R_0/k_s = 15$

Order	Pr or Sc	Re	m	R_n	A_n	Eigenvalues p_n^2
0	0.7	10000	0.2062	0.05036	8.789	55.66
1				-0.0995	-4.042	423.80
2				0.00237	1.514	2223.00
3				-0.00236	-2.075	2581.00
4				0.00100	1.102	6925.00
5				-0.00112	-1.492	6812.00
0	0.7	50000	0.0940	0.02305	19.360	198.50
1				-0.0451	-8.811	1527.40
2				0.00096	2.963	8923.00
3				-0.00105	-4.408	9545.00
4				0.00039	2.091	28670.00
5				-0.00048	-3.103	25730.00
0	0.7	100000	0.0667	0.01635	27.260	387.80
1				-0.00323	-12.520	2957.00
2				0.00067	4.072	17870.00
3				-0.00075	-6.264	18480.00
4				0.00027	2.861	57670.00
5				-0.00035	-4.410	49820.00

conductivity or diffusion, was given in accordance with the equations [23]:

$$f(\xi) = \frac{1}{2}\sqrt{1 + (2l^+)^2\tau^+} - 1,$$

where

$$\tau^+ = \frac{\tau}{\tau_{st}}, \tau_{st} = \lambda_* \frac{\rho \overline{w}^2}{8} = \frac{1}{2}\frac{dp}{d\eta}.$$

The turbulent kinematic viscosity $\nu_T(\xi)$ for rough channels was defined according to [23]. The mixing method was adopted in the form of Roth's equations [16] (here l is a mixing length):

$$l = 0; y < \Delta y;$$

$$l = \varkappa(y - \Delta y), y > \Delta y.$$

The velocity function $w(\xi)$ was given as the equations of a rod profile $w(\xi) = 1$ and the equations of logarithm profile of velocity in rough channels.

For rough channels the distribution of the functions $f(\xi)$ and $w(\xi)$ also depend not only on the distance from the wall y, the Reynolds and Prandtl (Schmidt) numbers, but also on the value of roughness height k_s and ratio k_s and distance y. It leads to the fact that the field of nominal dimensions for rough channels is more extensive, than for smooth channels. Therefore, the calculations were carried out mainly for the channels with significant roughness, as this case is of great interest for the subsequent analysis of the diffusion of heat and impurity in rough channels.

For some cases, the eigenvalues and constants of the solutions of the boundary value problems of heat and mass transfer for a cylindrical channel are given in Table 2.

The received results are qualitatively close to the results given in the previous section for the processes of heat and mass transfer in smooth pipes. However, the number of influencing factors in this case is greater. This is explained by the fact that the values of function $f(\xi)$ and its derivatives on the wall have significant effect on calculation results as the equations under consideration include value $f(1)$, which differs significantly for various models.

The setting of these values is rather complicated, since the experimental data are practically absent, and these values depend on the type of roughness and cannot be universal. In general, the carried-out calculations showed that the received solutions of boundary value problems are universal, since they allow to determine the fields of potentials and the Nusselt numbers both for smooth and rough channels. And the hydrodynamic and thermophysical features of the medium flow in the channel can be taken into account by a type of functions $f(\xi)$ and $w(\xi)$. The impact of the values of functions $f(\xi)$ and $w(\xi)$ in the boundary layer on the calculated indexes of heat and mass transfer is quite large. Finally, it should be noted that, if experimental values of the Nusselt number distribution through the length of the channel are known, it is possible to define the turbulent transfer parameters by comparing the Nusselt numbers with the obtained dependences.

6. Conclusion. On the basis of the executed calculations it can be concluded that the proposed method allows to obtain approximate solutions of a number of boundary value problems of turbulent heat and mass transfer. These solutions are quite simple and can be successfully used in calculations.

The universality and simplicity of the method is related to obtaining the solution in an analytical form, if we define transfer functions in general form, which allows to use different models of turbulent transfer in solutions.

The efficiency of the method is also ensured by the fact that for the first time it was possible to construct asymptotic approximations of coordinate functions (basis) in the process of solving and to find the residual of the functional for determining the constants and eigenvalues not in the Laplace transform domain, but in the original domain.

The reliability of a method was verified by comparison of calculation results, available experimental data and semi-empirical dependences for the condition of the stationary motions of media in smooth and rough cylindrical pipes.

Further research may be directed to the application of this method for solving the external problems of convective heat and mass transfer, where using the traditional approximate methods at half-finite intervals, such as the Bubnov-Galerkin method, is not always possible.

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